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Barnes-type multiple q-zeta functions and q-Euler polynomials*

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Abstract
The purpose of this paper is to present a systemic study of some families of multiple q-Euler numbers and polynomials and we construct multiple q-zeta functions which interpolate multiple q-Euler numbers at a negative integer. This is a partial answer to the open question in a previous publication (see Kim *et al* 2001 *J. Phys. A: Math. Gen.* **34** 7633–8).

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$.

Kurt Hensel (1861–1941) invented the so-called p-adic numbers around the end of the 19th century. Even though they have been discovered a 100 years ago, today these numbers are still enveloped in an aura of mystery in the science community. The p-adic numbers are used not only in mathematical physics, particularly in string theory and field theory, but also in other areas of natural sciences in which one faces complicated fractal behavior and hierarchical structures, for example, in turbulence theory, dynamical systems, statistical physics, biology, etc, cf [23–25, 30]. On the other hand, non-Archimedean functional analysis has been developed rapidly in recent years together with its applications in mathematical physics (see [1, 5–18, 23–26, 30]). Recently, theoretical physicists have envisioned ultrametric structures similar to tree-like structures arising in the study of physical systems, and have sought to construct related models using p-adic numbers and p-adic analysis. Spin glasses (with

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their ultrametric structures) formed an initial field of application of ultrametric techniques in theoretical physics (see [23–25]). In particular, the fact that at a very small distance, the physical space may no longer be Archimedean seems plausible to some mathematical physicists, cf [24–26, 30]. Therefore, p-adic analysis and the non-Archimedean geometry can be used not only for the description of geometry at small distances but also for describing the chaotic behavior of complicated systems, such as spin glasses and fractals, in the framework of traditional theoretical and mathematical physics, cf [23–26, 30]. In [27], Albeverio *et al* proved several limit theorems in the sense of the p-adic probability. The latter is understood as the p-adic limit of relative frequencies or, more generally, as a p-adic-valued measure. In 1995, Khrennikov considered the foundations of p-adic probability theory in which the probabilities of events are p-adic numbers. This theory is based on generalizations of the frequency theory of probability. The main idea is to consider the statistical stabilization of relative frequencies not only in the real topology but also in the p-adic topology (see [28]). A statistical interpretation of p-adic quantum mechanics and field theory with p-adic-valued functions is given by means of p-adic probability theory (see [28, 30]). There is an unexpected connection of p-adic analysis with q-analysis and quantum groups, and thus with noncommutative geometry q-analysis is a sort of q-deformation of the ordinary analysis. Spherical functions on quantum groups are q-special functions. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$, and if $q \in \mathbb{C}_p$, one normally assumes $|1-q|_p < 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q} \quad \text{and} \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \quad \text{(see [4–7])}.$$

The q-factorial is defined as $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$. For a fixed $d \in \mathbb{N}$ with $(p,d) = 1$, $d \equiv 1 (\text{mod } 2)$, we set

$$X = X_d = \lim_{\overleftarrow{N}} \mathbb{Z}/dp^N, \qquad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leqslant a < dp^N$. The q-binomial formulas are known as

$$(b;q)_n = (1-b)(1-bq)\ldots(1-bq^{n-1}) = \sum_{i=0}^{n} \binom{n}{i}_q q^{\binom{i}{2}}(-1)^i b^i$$

and

$$\frac{1}{(b;q)_n} = \frac{1}{(1-b)(1-bq)\ldots(1-bq^{n-1})} = \sum_{i=0}^{\infty} \binom{n+i-1}{i}_q b^i,$$

where $\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!}$ are the q-binomial coefficients for $n, k \in \mathbb{Z}_+$ (see [4, 8, 9]).

Recently, many authors have studied the q-extension in various areas (see [4–6]). In this paper, we consider the theory of q-integrals in the p-adic number field associated with Euler numbers and Euler polynomials closely related to the fermionic distribution. We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient $F_f(x,y) = \frac{f(x)-f(y)}{x-y}$ has a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,\mathrm{d}\mu_q(x) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \qquad \text{(see [7, 8, 9])}. \tag{1}$$

Thus, we note that

$$\lim_{q\to 1} I_q(f) = I_1(f) = \int_{\mathbb{Z}_p} f(x)\, \mathrm{d}\mu_1(x). \tag{2}$$

For $n \in \mathbb{N}$, let $f_n(x) = f(x+n)$. Then we have

$$I_1(f_n) = (-1)^n I_1(f) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l} f(l). \tag{3}$$

Using formula (3), we can readily derive the Euler polynomials, $E_n(x)$, namely

$$\int_{\mathbb{Z}_p} \mathrm{e}^{(x+y)t}\, \mathrm{d}\mu_1(y) = \frac{2}{\mathrm{e}^t+1}\,\mathrm{e}^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \qquad \text{(see [16–20])}.$$

In the special case $x = 0$, the sequence $E_n(0) = E_n$ is called the nth Euler number. In one of an impressive series of papers (see [1–3, 21, 23]), Barnes developed the so-called multiple zeta and multiple gamma functions. Barnes' multiple zeta function $\zeta_N(s, w|a_1,\ldots,a_N)$ depends on the parameters $a_1,\ldots,a_N$ that will be assumed to be positive. It is defined by the following series:

$$\zeta_N(s, w|a_1,\ldots,a_N) = \sum_{m_1,\ldots,m_N=0}^{\infty} (w + m_1a_1 + \cdots + m_Na_N)^{-s} \quad \text{for} \quad \Re(s) > N, \Re(w) > 0. \tag{4}$$

From (4), we can easily see that

$$\zeta_{M+1}(s, w + a_{M+1}|a_1,\ldots,a_{N+1}) - \zeta_{M+1}(s, w|a_1,\ldots,a_{N+1}) = -\zeta_M(s, w|a_1,\ldots,a_N),$$

and $\zeta_0(s, w) = w^{-s}$ (see [11]). Barnes showed that ζ_N had a mesomorphic continuation in s (with simple poles only at $s = 1, 2,\ldots,N$) and defined his multiple gamma function $\Gamma_N(w)$ in terms of the s-derivative at $s = 0$, which may be recalled here as follows: $\psi_n(w|a_1,\ldots,a_N) = \partial_s\zeta_N(s, w|a_1,\ldots,a_N)|_{s=0}$ (see [11]). Barnes' multiple Bernoulli polynomials $B_n(x, r|a_1,\ldots,a_r)$ are defined by

$$\frac{t^r}{\prod_{j=1}^r(\mathrm{e}^{a_jt}-1)}\,\mathrm{e}^{xt} = \sum_{n=0}^{\infty} B_n(x, r|a_1,\ldots,a_r)\frac{t^n}{n!}, \left(|t| < \max_{1\leqslant i\leqslant r}\frac{2\pi}{|a_i|}\right), \qquad \text{(see [11])}. \tag{5}$$

By (4) and (5), we see that

$$\zeta_N(-m, w|a_1,\ldots,a_N) = \frac{(-1)^N m!}{(N+m)!} B_{N+m}(w, N|a_1,\ldots,a_N),$$

where $w > 0$ and m is a positive integer.

By using the fermionic p-adic q-integral on $\mathbb{Z}_p$, we consider the Barnes-type multiple q-Euler polynomials and numbers in this paper. The main purpose of this paper is to present a systemic study of some families of Barnes-type multiple q-Euler polynomials and numbers. Finally, we construct a multiple q-zeta function which interpolates multiple q-Euler numbers at a negative integer. This is a partial answer to the open question in [6, p 7637].

Barnes' multiple zeta and gamma functions were also encountered by Shintani within the context of analytic number theory (see [11]). Recently, several mathematicians have studied multiple zeta functions or multiple zeta values and di-zeta values. It is apparent that these numbers are of interest and importance. In particular, Barnes' multiple zeta functions occur within the context of knot theory and quantum field theory, cf [10–16, 29, 30]. Meanwhile, the special values of Barnes' multiple zeta functions at a positive integer have come to the foreground in recent years both in connection with theoretical physics (Feynman diagrams) and the theory of mixed Tate motives, cf [30]. Ruijsenaars [29] showed how various known results concerning Barnes' multiple zeta and multiple gamma functions can be obtained as specializations of simple features shared by a quite extensive class of functions. The pertinent functions involve Laplace transforms, and their asymptotic behavior was obtained by exploiting them. Ruijsenaars demonstrated how Barnes' multiple zeta and multiple gamma functions fit into the recently developed theory of minimal solutions to the first-order analytic difference equations (see [29–31]). Both of these approaches to Barnes' functions gave rise to novel integral representations.

2. Barnes-type multiple q-Euler numbers and polynomials

Let $x, w_1, w_2, \ldots, w_r$ be the complex numbers with positive real parts. In $\mathbb{C}$, the Barnes-type multiple Euler numbers and polynomials are defined by

$$\frac{2^r}{\prod_{j=1}^{r}(\mathrm{e}^{w_j t}+1)}\,\mathrm{e}^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x|w_1,\ldots,w_r)\frac{t^n}{n!}, \quad \text{for} \quad |t| < \max\left\{\frac{\pi}{|w_i|}|i=1,\ldots,r\right\}, \tag{6}$$

and $E_n^{(r)}(w_1,\ldots,w_r) = E_n^{(r)}(0|w_1,\ldots,w_r)$, respectively (see [11, 12, 14]).

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. We first consider the q-extension of Euler polynomials as follows:

$$\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} \mathrm{e}^{[x+y]_q t}\mathrm{d}\mu_1(y) = 2\sum_{m=0}^{\infty}(-1)^m\,\mathrm{e}^{[m+x]_q t} \quad (\text{see [7, 8, 17]})\,. \tag{7}$$

Thus, we have $E_{n,q}(x) = \frac{2}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{1+q^l}$ (see [7]). In the special case $x = 0$, $E_{n,q} = E_{n,q}(0)$ is called the q-Euler number. The q-Euler polynomials of order $r \in \mathbb{N}$ are also defined by

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \mathrm{e}^{[x+x_1+\cdots+x_r]_q t}\mathrm{d}\mu_1(x_1)\cdots\mathrm{d}\mu_1(x_r)\\ &= 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m\mathrm{e}^{[m+x]_q t} \quad (\text{see [7, 8]}).\end{aligned} \tag{8}$$

In the special case $x = 0$, the sequence $E_{n,q}^{(r)}(0) = E_{n,q}^{(r)}$ is refereed as the q-extension of the Euler number of order r. Let $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$. Then we have

$$\begin{aligned}E_{n,q}^{(r)}(x) &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^n\,\mathrm{d}\mu_1(x_1)\ldots\mathrm{d}\mu_1(x_r)\\ &= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\sum_{a_1,\ldots,a_r=0}^{f-1}\ \sum_{m_1,\ldots,m_r=0}^{\infty} q^{l\{\sum_{i=1}^{r}(a_i+fm_i)\}}(-1)^{\sum_{i=1}^{r}(a_i+fm_i)}\\ &= 2^r\sum_{m_1,\ldots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n.\end{aligned} \tag{9}$$

By (8) and (9), we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$E_{n,q}^{(r)}(x) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n$$
$$= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m [m+x]_q^n.$$

Let $F_q^{(r)}(t,x) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x)\frac{t^n}{n!}$. Then we have

$$F_q^{(r)}(t,x) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m \, \mathrm{e}^{[m+x]_q t}$$
$$= 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \mathrm{e}^{[m_1+\cdots+m_r+x]_q t}. \tag{10}$$

Let χ be Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$. Then the generalized q-Euler polynomials attached to χ are defined by

$$\sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty} (-1)^m \chi(m)\, \mathrm{e}^{[m+x]_q t}. \tag{11}$$

Thus, we have

$$E_{n,\chi,q}(x) = \sum_{a=0}^{f-1} \chi(a)(-1)^a \int_{\mathbb{Z}_p} [x+a+fy]_q^n \, \mathrm{d}\mu_1(y)$$
$$= [f]_q^n \sum_{a=0}^{f-1} \chi(a)(-1)^a E_{n,q^f}\left(\frac{x+a}{f}\right). \tag{12}$$

In the special case, $x = 0$, the sequence $E_{n,\chi,q}(0) = E_{n,\chi,q}$ is called the nth generalized q-Euler numbers attached to χ. From (2) and (3), we can easily derive the following equation:

$$E_{m,\chi,q}(nf) - (-1)^n E_{m,\chi,q} = 2\sum_{l=0}^{nf-1} (-1)^{n-1-l}\chi(l)[l]_q^m.$$

Let us consider the higher order generalized q-Euler polynomials attached to χ as follows:

$$\int_X \cdots \int_X \left(\prod_{i=1}^{r} \chi(x_i)\right) \mathrm{e}^{[x_1+\cdots+x_r+x]_q t} \mathrm{d}\mu_1(x_1)\cdots \mathrm{d}\mu_1(x_r) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!}, \tag{13}$$

where $E_{n,\chi,q}^{(r)}(x)$ is called the nth generalized q-Euler polynomials of order r attached to χ. By (13), we see that

$$E_{n,\chi,q}^{(r)}(x) = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx}(-1)^l \sum_{a_1,\ldots,a_r=0}^{f-1} \left(\prod_{j=1}^{r} \chi(a_j)\right) \frac{(-q^l)^{\sum_{i=1}^r a_i}}{(1+q^{lf})^r}$$
$$= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m \sum_{a_1,\ldots,a_r=0}^{f-1} \left(\prod_{j=1}^{r} \chi(a_j)\right) (-1)^{\sum_{i=1}^r a_i} \left[\sum_{j=1}^{r} a_j + x + mf\right]_q^n, \tag{14}$$

and

$$\sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{\sum_{j=1}^r m_j} \left(\prod_{i=1}^{r} \chi(m_i)\right) \mathrm{e}^{[x+\sum_{j=1}^r m_j]_q t}. \tag{15}$$

In the special case, $x = 0$, the sequence $E^{(r)}_{n,\chi,q}(0) = E^{(r)}_{n,\chi,q}$ is called the nth generalized q-Euler numbers of order r attached to χ.

By (14) and (15), we obtain the following theorem.

Theorem 2. *Let χ be Dirichlet's character with conductor $f \in \mathbb{N}$, with $f \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, $r \in \mathbb{N}$, we have*

$$E^{(r)}_{n,\chi,q}(x) = \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}q^{lx}(-1)^l\sum_{a_1,\ldots,a_r=0}^{f-1}\left(\prod_{j=1}^{r}\chi(a_j)\right)\frac{(-q^l)^{\sum_{i=1}^r a_i}}{(1+q^{lf})^r}$$
$$= 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^m\sum_{a_1,\ldots,a_r=0}^{f-1}\left(\prod_{j=1}^{r}\chi(a_j)\right)(-1)^{\sum_{i=1}^r a_i}\left[\sum_{j=1}^{r}a_j + x + mf\right]_q^n$$
$$= 2^r\sum_{m_1,\ldots,m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}\left(\prod_{i=1}^{r}\chi(m_i)\right)[x+m_1+\cdots+m_r]_q^n.$$

For $h \in \mathbb{Z}$ and $r \in \mathbb{N}$, we introduce the extended higher order q-Euler polynomials as follows:

$$E^{(h,r)}_{n,q}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{j=1}^r (h-j)x_j}[x+x_1+\cdots+x_r]_q^n\,\mathrm{d}\mu_1(x_1)\ldots\mathrm{d}\mu_1(x_r) \quad \text{(see [8])}. \tag{16}$$

From (16), we note that

$$E^{(h,r)}_{n,q}(x) = 2^r\sum_{m_1,\ldots,m_r=0}^{\infty} q^{(h-1)m_1+\cdots+(h-r)m_r}(-1)^{m_1+\cdots+m_r}[x+m_1+\cdots+m_r]_q^n. \tag{17}$$

It is known in [8] that

$$E^{(h,r)}_{n,q}(x) = \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-q^x)^l}{(-q^{h-r+l};q)_r} = 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-q^{h-r})^m[x+m]_q^n. \tag{18}$$

Let $F^{(h,r)}_q(t,x) = \sum_{n=0}^{\infty}E^{(h,r)}_{n,q}(x)\frac{t^n}{n!}$. Then we have

$$F^{(h,r)}_q(t,x) = 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q q^{(h-r)m}(-1)^m\,\mathrm{e}^{[m+x]_q t}$$
$$= 2^r\sum_{m_1,\ldots,m_r=0}^{\infty} q^{\sum_{j=1}^r (h-j)m_j}(-1)^{\sum_{j=1}^r m_j}\mathrm{e}^{[x+m_1+\cdots+m_r]_q t}. \tag{19}$$

Therefore, we obtain the following theorem.

Theorem 3. *For $h \in \mathbb{Z}$, $r \in \mathbb{N}$ and $x \in \mathbb{Q}^+$, we have*

$$E^{(h,r)}_{n,q}(x) = 2^r\sum_{m_1,\ldots,m_r=0}^{\infty} q^{(h-1)m_1+\cdots+(h-r)m_r}(-1)^{m_1+\cdots+m_r}[m_1+\cdots+m_r+x]_q^n.$$

For $f \in \mathbb{N}$, with $f \equiv 1 \pmod 2$, it is easy to show the following distribution relation for $E^{(h,r)}_{n,q}(x)$:

$$E^{(h,r)}_{n,q}(x) = [f]_q^n\sum_{a_1,\ldots,a_r=0}^{f-1}(-1)^{a_1+\cdots+a_r}q^{\sum_{j=1}^r (h-j)a_j}E_{n,q^f}\left(\frac{x+a_1+\cdots+a_r}{f}\right).$$

Let us consider Barnes-type multiple q-Euler polynomials. For $w_1, \ldots, w_r \in \mathbb{Z}_p$, we define the Barnes-type q-multiple Euler polynomials as follows:

$$E_{n,q}^{(r)}(x|w_1, \ldots, w_r) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\sum_{j=1}^{r} w_j x_j + x\right]_q^n \mathrm{d}\mu_1(x_1) \ldots \mathrm{d}\mu_1(x_r). \tag{20}$$

From (20), we can easily derive the following equation:

$$E_{n,q}^{(r)}(x|w_1, \ldots, w_r) = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-q^x)^l}{(1+q^{lw_1}) \ldots (1+q^{lw_r})}, \quad \text{(see [8])}. \tag{21}$$

Thus, we have

$$E_{n,q}^{(r)}(x|w_1, \ldots, w_r) = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-q^x)^l \sum_{a_1,\ldots,a_r=0}^{f-1} \frac{(-1)^{\sum_{i=1}^{r} a_i} q^{l \sum_{j=1}^{r} w_j a_j}}{(1+q^{lfw_1}) \ldots (1+q^{lfw_r})}, \tag{22}$$

where $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$. By (22), we see that

$$E_{n,q}^{(r)}(x|w_1, \ldots, w_r) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} [x + w_1 m_1 + \cdots + w_r m_r]_q^n. \tag{23}$$

In the special case, $x = 0$, the sequence $E_{n,q}^{(r)}(w_1, \ldots, w_r) = E_{n,q}^{(r)}(0|w_1, \ldots, w_r)$ is called the nth Barnes-type multiple q-Euler number. Let $F_q^{(r)}(t, x|w_1, \ldots, w_r) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|w_1, \ldots, w_r) \frac{t^n}{n!}$. Then we have

$$F_q^{(r)}(t, x|w_1, \ldots, w_r) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} \mathrm{e}^{[x+w_1m_1+\cdots+w_rm_r]_q t}. \tag{24}$$

Therefore, we obtain the following theorem.

Theorem 4. *For $w_1, \ldots, w_r \in \mathbb{Z}_p$, $r \in \mathbb{N}$ and $x \in \mathbb{Q}^+$, we have*

$$\begin{aligned} E_{n,q}^{(r)}(x|w_1, \ldots, w_r) &= 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} [x + m_1 w_1 + \cdots + m_r w_r]_q^n \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-q^x)^l}{(1+q^{lw_1}) \cdots (1+q^{lw_r})}. \end{aligned}$$

For $w_1, \ldots, w_r \in \mathbb{Z}_p$, $a_1, \ldots, a_r \in \mathbb{Z}$, we consider another q-extension of Barnes-type multiple q-Euler polynomials as follows:

$$\begin{aligned} &E_{n,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[x + \sum_{j=1}^{r} w_j x_j\right]_q^n q^{\sum_{i=1}^{r} a_i x_i} \mathrm{d}\mu_1(x_1) \cdots \mathrm{d}\mu_1(x_r). \end{aligned} \tag{25}$$

Thus, we have

$$E_{n,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r) = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{(1+q^{lw_1+a_1}) \cdots (1+q^{lw_r+a_r})}. \tag{26}$$

From (25) and (26), we can derive the following equation:

$$E_{n,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r) = 2^r \sum_{m_1,\ldots,m_r=0} (-1)^{\sum_{j=1}^{r} m_j} q^{\sum_{i=1}^{r} a_i m_i} \left[x + \sum_{j=1}^{r} w_j x_j\right]_q^n. \tag{27}$$

Let $F_q^{(r)}(t, x|w_1, \ldots, w_r; a_1, \ldots, a_r) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r)\frac{t^n}{n!}$. Then, we have

$$F_q^{(r)}(t, x|w_1, \ldots, w_r; a_1, \ldots, a_r) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} q^{a_1m_1+\cdots+a_rm_r} \mathrm{e}^{[x+w_1m_1+\cdots+w_rm_r]_q t}. \tag{28}$$

Theorem 5. *For $r \in \mathbb{N}$, $w_1, \ldots, w_r \in \mathbb{Z}_p$ and $a_1, \ldots, a_r \in \mathbb{Z}$, we have*

$$E_{n,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots a_r) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{\sum_{j=1}^r m_j} q^{\sum_{i=1}^r a_i m_i} \left[x + \sum_{j=1}^r w_j m_j\right]_q^n.$$

Let χ be Dirichlet's character with conductor $f \in \mathbb{N}$, with $f \equiv 1 \pmod 2$. Now we consider the generalized the Barnes-type q-multiple Euler polynomials attached to χ as follows:

$$E_{n,\chi,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r) = \int_X \cdots \int_X [x + w_1x_1 + \cdots + w_rx_r]_q^n \left(\prod_{j=1}^r \chi(x_j)\right) q^{a_1x_1+\cdots+a_rx_r} \mathrm{d}\mu_1(x_1)\ldots \mathrm{d}\mu_1(x_r).$$

Thus, we have

$$E_{n,\chi,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r) = \frac{2^r}{(1-q)^n} \sum_{b_1,\ldots,b_r=0}^{f-1} \left(\prod_{i=1}^r \chi(b_i)\right) (-1)^{\sum_{j=1}^r b_j} q^{\sum_{i=1}^r (lw_i+a_i)b_i} \frac{\sum_{l=0}^n \binom{n}{l}(-1)^l q^{lx}}{\prod_{j=1}^r (1+q^{(lw_j+a_j)f})}. \tag{29}$$

From (29), we note that

$$E_{n,\chi,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{j=1}^r \chi(m_i)\right) (-1)^{m_1+\cdots+m_r} q^{a_1m_1+\cdots+a_rm_r} \left[x + \sum_{j=1}^r w_j m_j\right]_q^n.$$

Therefore, we obtain the following theorem.

Theorem 6. *For $r \in \mathbb{N}$, $w_1, \ldots, w_r \in \mathbb{Z}_p$ and $a_1, \ldots, a_r \in \mathbb{Z}$, we have*

$$E_{n,\chi,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{j=1}^r \chi(m_i)\right) (-1)^{m_1+\cdots+m_r} q^{a_1m_1+\cdots+a_rm_r} \left[x + \sum_{j=1}^r w_j m_j\right]_q^n.$$

Let $F_{q,\chi}^{(r)}(t, x|w_1, \ldots, w_r; a_1, \ldots, a_r) = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r)\frac{t^n}{n!}$.
By theorem 6, we see that

$$F_{q,\chi}^{(r)}(t, x|w_1, \ldots, w_r; a_1, \ldots, a_r) = 2^r \sum_{m_1,\cdots,m_r=0}^{\infty} \left(\prod_{j=1}^r \chi(m_i)\right) (-1)^{m_1+\cdots+m_r} q^{a_1m_1+\cdots+a_rm_r} \mathrm{e}^{[x+\sum_{j=1}^r w_jm_j]_q t}. \tag{30}$$

3. Barnes-type multiple q-zeta functions

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$ and the parameters $w_1, \ldots, w_r$ are positive. From (28), we consider the Barnes-type multiple q-Euler polynomials in $\mathbb{C}$ as follows:

$$\begin{aligned} &F_q^{(r)}(t, x|w_1, \ldots, w_r; a_1, \ldots, a_r) \\ &\quad = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-1)^{m_1+\cdots+m_r} q^{a_1m_1+\cdots+a_rm_r} \, \mathrm{e}^{[x+w_1m_1+\cdots+w_rm_r]_q t} \\ &\quad = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r)\frac{t^n}{n!}, \quad \text{for} \quad |t| < \max_{1\leqslant i\leqslant r}\left\{\frac{\pi}{|w_i|}\right\}. \end{aligned} \tag{31}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0, a_1, \ldots, a_r \in \mathbb{C}$, we can derive the following equation (32) from the Mellin transformation of $F_q^{(r)}(t, x|w_1, \ldots, w_r; a_1, \ldots, a_r)$:

$$\begin{aligned} &\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1} F_q^{(r)}(-t, x|w_1, \ldots, w_r; a_1, \ldots, a_r)\mathrm{d}t \\ &\quad = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{m_1a_1+\cdots+m_ra_r}}{[x+w_1m_1+\cdots+w_rm_r]_q^s}. \end{aligned} \tag{32}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, $a_1, \ldots, a_r \in \mathbb{C}$, we define the Barnes-type multiple q-zeta function as follows:

$$\zeta_{q,r}(s, x|w_1, \ldots, w_r; a_1, \ldots, a_r) = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_r} q^{m_1a_1+\cdots+m_ra_r}}{[x+w_1m_1+\cdots+w_rm_r]_q^s}. \tag{33}$$

Note that $\zeta_{q,r}(s, x|w_1, \ldots, w_r)$ is the meromorphic function in the whole complex s-plane. By using the Mellin transformation and the Cauchy residue theorem, we obtain the following theorem which is a part of answer to the open question in [6, p 7637].

Theorem 7. *For $x \in \mathbb{C}$ with $\Re(x) > 0$, $n \in \mathbb{Z}_+$, we have*

$$\zeta_{q,r}(-n, x|w_1, \ldots, w_r; a_1, \ldots, a_r) = E_{n,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r).$$

Let χ be Dirichlet's character with conductor $f \in \mathbb{N}$ with $f \equiv 1 \pmod 2$. From (30), we can define the generalized Barnes-type multiple q-Euler polynomials attached to χ in $\mathbb{C}$ as follows:

$$\begin{aligned} &F_{q,\chi}^{(r)}(t, x|w_1, \ldots, w_r; a_1, \ldots, a_r) \\ &\quad = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \left(\prod_{j=1}^{r} \chi(m_i)\right) (-1)^{m_1+\cdots+m_r} q^{a_1m_1+\cdots+a_rm_r} \, \mathrm{e}^{[x+\sum_{j=1}^r w_jm_j]_q t} \\ &\quad = \sum_{n=0}^{\infty} E_{n,\chi,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r)\frac{t^n}{n!}. \end{aligned} \tag{34}$$

From (34) and the Mellin transformation of $F_{q,\chi}^{(r)}(t, x|w_1, \ldots, w_r; a_1, \ldots, a_r)$, we can easily derive the following equation (35) :

$$\begin{aligned} &\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1} F_{q,\chi}^{(r)}(-t, x|w_1, \ldots, w_r; a_1, \ldots, a_r)\,\mathrm{d}t \\ &\quad = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{\left(\prod_{j=1}^{r} \chi(m_i)\right)(-1)^{m_1+\cdots+m_r} q^{m_1a_1+\cdots+m_ra_r}}{[x+w_1m_1+\cdots+w_rm_r]_q^s}. \end{aligned} \tag{35}$$

For $s, x \in \mathbb{C}$ with $\Re(x) > 0$, we also define the Barnes-type multiple q-l-function as follows:

$$l_{q,\chi}^{(r)}(s, x|w_1, \ldots, w_r; a_1, \ldots, a_r)$$
$$= 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{\left(\prod_{j=1}^{r} \chi(m_j)\right)(-1)^{m_1+\cdots+m_r} q^{m_1 a_1+\cdots+m_r a_r}}{[x + w_1 m_1 + \cdots + w_r m_r]_q^s}. \tag{36}$$

Note that $l_{q,\chi}^{(r)}(s, x|w_1, \ldots, w_r)$ is the meromorphic function in the whole complex s-plane. By using (34), (35), (36) and the Cauchy residue theorem, we obtain the following theorem.

Theorem 8. *For $x, s \in \mathbb{C}$ with $\Re(x) > 0$, $n \in \mathbb{Z}_+$, we have $l_{q,\chi}^{(r)}(-n, x|w_1, \ldots, w_r; a_1, \ldots, a_r) = E_{n,\chi,q}^{(r)}(x|w_1, \ldots, w_r; a_1, \ldots, a_r)$.*

We note that theorem 8 is r-multiplication of Dirichlet's-type q–l-series. Theorem 8 seems to be interesting and worth studing in the area of multiple p-adic l-function or mathematical physics related to Knot theory and ζ-function (see [4–22]).

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